PARAMETER ESTIMATION OF THE BRENNAN-SCHWARTZ MODEL

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Abstract. The Brennan-Schwartz model is one of the stochastic differential equation models for the interest-rate under the risk-neutral probability measure. It requires estimation of parameters whose values are unknown. However, data are collected in the real world, and their statistical properties characterize the distribution of interest-rate process under the actual probability measure. In this paper, the method which is used to estimate parameters of the Brennan-Schwartz model is the Maximum Likelihood estimation method which maximizes the likelihood ratio. It will be justified heuristically by approximating this likelihood ratio with the Euler difference scheme. However, it cannot be analytically solved. Therefore, in this paper, the estimation will be continued iteratively using the Nelder-Mead Algorithm. Finally, the result will be compared with Maximum Likelihood estimation which changing the measure of the Brennan-Schwartz model as obtained in [4].

Keywords: Brennan-Schwartz model, maximum likelihood, Radon-Nikodym derivative

1. Introduction

The Brennan-Schwartz model is one of the stochastic differential equation models for the interest-rate under the risk-neutral probability measure. It is the short-rate model because the interest-rate process is the interest-rate for short-term borrowing. Since this interest-rate is determined by only one stochastic differential equation, the Brennan-Schwartz model is said to have one factor. The primary shortcoming of one-factor models is that they cannot capture complicated yield curve behavior and they tend to produce parallel shifts in the yield curve but not changes in its slope or curvature [15]. Like the Vasicek, the Rendleman-Bartter, or the Cox-Ingersoll-Ross model, the Brennan-Schwartz model is also belong to the equilibrium model that has the properties of mean reversion. It is similar to the Vasicek model, but the diffusion coefficient is multiplicative.

The Brennan-Schwartz model requires estimation of parameters whose values are unknown. However, data are collected in the real world, and their statistical properties characterize the distribution of interest-rate process under the actual probability measure [3]. Therefore, we need to change of measure of Brennan-Schwartz model. However, the Brennan-Schwartz model under the actual probability measure depends on the new parameter which contributing to the market price of risk and difficult to be solved as obtained in [4]. This new parameter estimation problem can be eliminated by applying likelihood ratio concept [11] without changing the measure of the Brennan-Schwartz model. In this paper, the method which is used to estimate parameters of the Brennan-Schwartz model is the Maximum Likelihood estimation method which maximizes the likelihood ratio.

The organization of the paper is as follows. In Section 2 the Brennan-Schwartz model is introduced with analytical solution and its stability criteria is discussed. Section 3 considers the Maximum Likelihood procedure to estimate parameters of the Brennan-Schwartz model which be implemented to approximate the real data of monthly interest-rate from a zero-coupon bond with maturity time of 5 years: January, 1982 - February, 2011 in which data is downloaded from www.bankofengland.co.uk. Finally, the result will be compared with
Maximum Likelihood estimation which changing the measure of the Brennan-Schwartz model as obtained in [4], and we conclude in Section 4.

II. Model

The Brennan-Schwartz model is one of the stochastic differential equation models that represent the dynamics of the interest-rate as follows [2]:

\[ dr_t = \gamma (x - r_t) dt + \beta r_t d\tilde{W}_t, \quad (2.1) \]

where \( \gamma, \alpha, \beta \) are nonnegative constants [16] and \( \tilde{W}_t \) is a Brownian motion under the risk-neutral probability measure \( \tilde{P} \) [15]. \( \alpha \) is defined as a reversion level of the rate of interest \( r_t \), \( \gamma \) as a speed of adjustment, and \( \beta r_t^2 \) as the infinitesimal variance of the process. The quantity \( \gamma (\alpha - r_t) \) and \( \beta r_t \) are called the drift and diffusion coefficient of the process, respectively. It is seen that the drift coefficient depends on the current value of \( r_t \). If \( r_t \) is less than \( \alpha \), then the drift coefficient is positive; if \( r_t \) is greater than \( \alpha \), then it is negative. Thus, the interest-rate \( r_t \) tends to revert to its reversion level \( \alpha \) at a rate which depends upon the speed of adjustment \( \gamma \) [2]. The mean reversion property can be illustrated in Figure 2.1 [5].

![Figure 2.1 Mean reversion property](image)

Based on the existence and uniqueness theorem [10], the Brennan-Schwartz model satisfies three regularity conditions:

1. \( |\gamma (\alpha - r_t) - \gamma (\alpha - r_s)| + |\beta r_t - \beta r_s| \leq C |r_t - r_s| \) for some constant \( C \),

2. \( |\gamma (\alpha - r_t)| + |\beta r_t| \leq D (1 + |r_t|) \) for some constant \( D \),

3. \( E[|r_t|^2] < \infty \).

Hence, it has a unique solution which can be solved explicitly [4] by applying Ito-Doeblin formula and Ito product rule [9, 15] as follows:

\[ r_t = r_0 \cdot \exp \left\{ - \left( \gamma + \frac{1}{2} \beta^2 \right) t + \beta \tilde{W}_t \right\} + \gamma \alpha \int_0^t \exp \left\{ - \left( \gamma + \frac{1}{2} \beta^2 \right) (t - u) + \beta (\tilde{W}_u - \tilde{W}_s) \right\} du. \quad (2.2) \]

From the existence and uniqueness theory we know that the solutions of a differential equation are continuous in their initial values, at least over a finite time interval. Extending this idea to an infinite time interval leads to the concept of stability [6]. Stability is important to describe resistance of the model to the perturbation in the initial state or parameters of the model [1]. Two ways to define stochastic stability will be considered in this paper, stochastically asymptotically stable and mean-square stability [6]. Their criteria for the Brennan-Schwartz
model are \( \gamma + \frac{1}{2} \beta^2 > 0 \) and \( 2\gamma - \beta^2 > 0 \), respectively [4]. These stability criteria can be used as guidelines for selecting parameter that makes the model becomes resistant to the perturbation. However, the Brennan-Schwartz model requires estimation of parameters whose values are unknown.

In the next section, parameters of the Brennan-Schwartz model will be estimated by the Maximum Likelihood estimation method which maximizes the likelihood ratio. It can be constructed via the Girsanov theorem with a Radon-Nikodym derivative of the relevant probability measures [12, 13]. The Radon-Nikodym derivative produces the relevant probability density and can be regarded as a change of measure among the absolutely continuous probability measures [13]. It can be justified heuristically by approximating this likelihood ratio with the Euler difference scheme [6].

3. Results

The Maximum Likelihood estimation method for a stochastic differential equation model needs a transition density. However, the transition density of the Brennan-Schwartz model does not have a closed-form analytic expression [4]. As a result, the exact Maximum Likelihood method is not generally applicable. If a continuous trajectory of the solution process \( r_t \) were recorded over the time interval \( 0 \leq t \leq T \), direct Maximum Likelihood estimation would be possible based on the continuous path likelihood. While the Brennan-Schwartz model as in (2.1) is formulated in continuous time, the sample data are always collected at discrete points in time or over discrete intervals in the case of flow data. To address this complication, Euler difference scheme approach has been developed involves approximating the likelihood function. Under certain conditions [7] the maximum likelihood estimate parameters of Brennan-Schwartz model \( \hat{\gamma}, \hat{\alpha}, \) and \( \hat{\beta} \) determined from continuous observation of a trajectory of a solution process \( r_t \) over the time interval \( 0 \leq t \leq T \) is the values of \( \gamma, \alpha, \) and \( \beta \), respectively, which maximizes the likelihood ratio

\[
\ell(\gamma, \alpha, \beta) = \int_0^T \frac{\gamma (\alpha - r_t)}{(\beta r_t)^2} dt - \frac{1}{2} \int_0^T \frac{(\gamma (\alpha - r_t))^2}{(\beta r_t)^2} dt. \tag{3.1}
\]

The integrals in the equation (3.1) can be approximated by Euler difference scheme approach [6]. It is assumed that \( r_{t_k}, r_{t_{k+1}}, \ldots, r_{t_N} \) are observed values of \( r_t \), \( 0 \leq t \leq T \), at the respective uniformly distributed times \( t_k = kT/N \) for \( k = 0,1,2,\ldots,N \). Letting \( r_{t_k} = r_k \) at \( t = t_k \), then we obtain the Euler difference scheme for Brennan-Schwartz model (2.1)

\[
r_{t_{k+1}} = r_k + \gamma (\alpha - r_k) \Delta + \beta r_k \Delta W_{t_k},
\]

for \( k = 0,1,\ldots,N-1 \) where \( \Delta = T/N \). The increments \( \Delta W_{t_k} = \bar{W}_{t_k} - \bar{W}_{t_{k-1}}, \Delta W_{t_{k+1}} = \bar{W}_{t_{k+1}} - \bar{W}_{t_k} \) of the Brownian motion are independent \( \mathcal{N}(0, \Delta) \) distributed random variables. Writing \( \Delta r_k = r_{t_{k+1}} - r_k \), we obtain the approximate log-likelihood function (3.1)

\[
\ell(\gamma, \alpha, \beta) = \sum_{k=0}^{N-1} \frac{\gamma (\alpha - r_k) \Delta r_k}{(\beta r_k)^2} - \frac{1}{2} \sum_{k=0}^{N-1} \frac{(\gamma (\alpha - r_k))^2 \Delta}{(\beta r_k)^2}
\]

\[
= \sum_{k=0}^{N-1} \frac{\gamma (\alpha - r_k) \Delta r_k - (\gamma (\alpha - r_k))^2 \Delta}{2(\beta r_k)^2}.
\]
When sampling interval $\Delta$ is small, the Euler scheme should provide a good approximation to the exact discrete time model. However, when it is large, the Euler approximation can be poor. The Euler discretization offers a good approximation to the exact discrete time model for daily or higher frequencies but not for annual or lower frequencies. The advantages of the Euler method include the ease with which the likelihood function is obtained, the low computational cost, and the wide range of its applicability [13]. The biggest problem with the procedure is that when sampling interval is fixed the estimator is inconsistent [8].

However, the equation (3.2) cannot be analytically solved. Therefore, in this paper, the estimation will be continued iteratively using the Nelder-Mead Algorithm [14] which minimizing the $-\ell(\gamma, \alpha, \beta)$. Next, the Brennan-Schwartz model will be applied to approximate the real data of monthly interest-rate from a zero-coupon bond with maturity time of 5 years: January, 1982 - February, 2011 in which data is downloaded from www.bankofengland.co.uk, by using both of parameter estimators which obtained from the Maximum Likelihood procedure in this paper and in [4]. In this paper, Matlab software is used to help calculation in the estimation problem and to illustrate the interest-rate movement in the approximation problem. Hence, parameter estimators which belong to the stochastically asymptotically and mean square stability criteria of Brennan-Schwartz model are as follow:

<table>
<thead>
<tr>
<th>Parameter estimators (with changing the measure of the model [4])</th>
<th>Parameter estimators (without changing the measure of the model)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\gamma} = 0.135$</td>
<td>$\hat{\gamma} = 0.0621$</td>
</tr>
<tr>
<td>$\hat{\alpha} = 0.04148$</td>
<td>$\hat{\alpha} = 0.0117$</td>
</tr>
<tr>
<td>$\hat{\beta} = 0.13$</td>
<td>$\hat{\beta} = 0.0563$</td>
</tr>
</tbody>
</table>

Figure below visualized the approximation of interest-rate movement of the solution of Brennan-Schwartz model (2.2) by using parameter estimators which obtained from the Maximum Likelihood procedure with and without changing the measure of the model.

![Figure 3.1 Interest-rate movements of Brennan-Schwartz model](image)

The norm error which is the maximum absolute error for Brennan-Schwartz model by using parameter estimators which obtained from the Maximum Likelihood procedure with and without changing the measure of the model are 3.68% and 3.85%, respectively. To approximate the real data, it is seen that the interest-rate movements of Brennan-Schwartz model by using parameter estimators which obtained from the Maximum Likelihood procedure with and without changing the measure of the model have similar movements.
4. Conclusion
In this paper, parameters of the Brennan-Schwartz model can be estimated by using the Maximum Likelihood estimation method which maximizes the likelihood ratio without changing the measure of the model as obtained in [4]. However, the transition density of the Brennan-Schwartz model does not have a closed-form analytic expression and the sample data which collected in discrete time are different with the Brennan-Schwartz model which formulated in continuous time. To address this complication, Euler difference scheme approach is applied in approximating the likelihood ratio. Hence, the Brennan-Schwartz model by using parameter estimators which obtained from the Maximum Likelihood procedure with and without changing the measure of the model have similar movements.

References